Set Theory Definitions

Set Membership, Equality, and Subsets

An element of a set is an object directly contained within that set. For example, $1 \in \{1, 2, 3\}$ and $\emptyset \in \{\emptyset\}$, but $1 \notin \emptyset$ and $1 \notin \{2, 3\}$. Note that $1 \notin \{\{1\}\}$, and $\{1\} \notin \{1\}$, but $\{1\} \in \{\{1\}\}$.

Two sets are equal if they contai the same elements. For example, we have that $\{1, 2\} = \{2, 1\}$ and that $\{ \emptyset \} = \{ \emptyset \}$. However, $\{ \emptyset \} \neq \{ \{\emptyset\} \}$, because each set contains an element the other does not. A set and a non-set are never equal; in particular, this means $x \neq \{x\}$ for any x.

A set A is a subset of a set B (denoted $A \subseteq B$) if every element of A is also an element of B:

$$\mathbb{N} \subseteq \mathbb{Z} \qquad \{1, 2, 3\} \subseteq \{1, 2, 3, 4\} \qquad \{1\} \subseteq \{1, \{1\}, \{\{1\}\}\}$$

Set Operations

The set $\{ x \mid \textit{some property of } x \}$ is the set of all x's satisfying the given property. Formally, we have that $w \in \{ x \mid \textit{some property of } x \}$ if and only if the specified property holds for w.

The set $A \cup B$ is the set $\{ x \mid x \in A \text{ or } x \in B \}$. Equivalently, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

The set $A \cap B$ is the set $\{ x \mid x \in A \text{ and } x \in B \}$. Equivalently, $x \in A \cap B$ precisely if $x \in A$ and $x \in B$.

The set $A - B$ is the set $\{ x \mid x \in A \text{ and } x \notin B \}$. This set is also sometimes denoted $A \setminus B$.

The set $A \Delta B$ is the set $\{ x \mid \text{exactly one of } x \in A \text{ and } x \in B \text{ is true} \}$.

Power Sets

The power set of a set S, denoted $\wp(S)$, is the set of all subsets of S. Using set-builder notation, this is the set $\wp(S) = \{ U \mid U \subseteq S \}$. Cantor's Theorem states that $|S| < |\wp(S)|$ for every set S.

Special Sets

The set $\emptyset = \{ \}$ is the empty set containing no elements.

The set $\mathbb{N} = \{0, 1, 2, 3, 4, \dots \}$ is the set of all natural numbers. We treat 0 as a natural number.

The set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of all integers.

The set $\mathbb{R}$ consists of all the real numbers. The set $\mathbb{Q}$ consists of all rational numbers.

Cardinality

The cardinality of a finite set S (denoted $|S|$) is the natural number equal to the number of elements in that set. The cardinality of $\mathbb{N}$ (denoted $|\mathbb{N}|$) is $\aleph_0$ (pronounced “aleph-nought”). Two sets have the same cardinality if there is a way of pairing up each element of the two sets such that every element of each set is paired with exactly one element of the other set.